D1-82-0161

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Relationship Between System Failure Rate and Component Failure Rates

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Mathematics Research

April 1962

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Mathematical Note No. 256

Mathematics Research Laboratory

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1. Introduction A number of interesting and important consequences follow once we know that a system has an increasing failure rate. For example, we can obtain moment inequalities and bounds for system life. As shown in [1] we can infer that under certain assumptions successive intervals between system checks should form a decreasing sequence. Also we can compute more readily the optimum policy in a certain planned replacement model [3]. Clearly it would be valuable to have simple sufficient conditions under which a system of components each having increasing failure rate itself has an increasing failure rate. In this paper we obtain such a condition for systems of like components; for systems of components having differing reliabilities we obtain upper and lower bounds which are increasing functions. In addition we obtain more general relationships between system failure rate and component failure rates.

2. Basic Theorem for Systems of Identical Components In this section, we assume that the structure consists of independent like components, with each component life distributed according to the common probability distribution $F(t)$. If at a given instant of time each component has reliability $p = 1 - F(t)$, then the system reliability will be designated by $h(p)$ [4].

The main result of this section is

Theorem 1 Assume a structure with reliability function $h(p)$, with each component life independently distributed according to distribution F having density f. Then

(a) $$\frac{ph'(p)}{h(p)}\bigg|_{p=1-F(t)} = \frac{R(t)}{r(t)} ,$$

where $r(t) = \dfrac{f(t)}{1 - F(t)}$ = component failure rate at time t, and $R(t)$ = system failure rate at time t.

(b) $\dfrac{ph'(p)}{h(p)}$ is a decreasing function of p if and only if $\dfrac{R(t)}{r(t)}$ is an increasing function of t;

(c) if $r(t)$ is an increasing function of t and $\dfrac{ph'(p)}{h(p)}$ is a decreasing function of p, then $R(t)$ is an increasing function of t.

Thus result (c) gives a simple sufficient condition on system structure which will preserve a monotone failure rate when a system is constructed out of independent like components. We shall present an important class of structures which satisfy this sufficient condition.

To prove (a), let $S(t)$ represent the probability of system survival past time t; i.e., $S(t) = h(1 - F(t))$. By definition

$$R(t) = \frac{-S'(t)}{S(t)} = \frac{h'(p)}{h(p)}\bigg|_{p=1-F(t)} \cdot f(t) = \frac{ph'(p)}{h(p)}\bigg|_{p=1-F(t)} \cdot \frac{f(t)}{1 - F(t)} ,$$

so that

$$\frac{R(t)}{r(t)} = \frac{ph'(p)}{h(p)}\bigg|_{p=1-F(t)} ,$$

establishing (a).

To prove (b), simply note that $p = 1 - F(t)$ is a decreasing function of t.

Finally, (c) is an immediate consequence of (b) and the fact that p is a decreasing function of t. ||

An important class of systems for which the condition $\frac{ph'(p)}{h(p)}$ is a decreasing function of p are the so-called k out of n structures. Birnbaum, Esary, and Saunders [4, p. 58], define a k out of n structure as one that performs if at least k of its n components perform, and which does not perform otherwise. Note that the 1 out of n structure is the well-known parallel structure and the n out of n structure is the well-known series structure. [7] proves that a k out of n structure consisting of n independent components has a ratio $\frac{ph'(p)}{h(p)}$ decreasing in p; we give the following proof. Write

$$h(p) = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n+1-k)} \int_0^p t^{k-1}(1-t)^{n-k} dt,$$

the incomplete Beta function [5, pp. 83-87]. Thus

$$\frac{h(p)}{ph'(p)} = \frac{1}{p} \int_0^p \left(\frac{t}{p}\right)^{k-1} \left(\frac{1-t}{1-p}\right)^{n-k} dt.$$

Letting $u = \frac{t}{p}$, we have

$$\frac{h(p)}{ph'(p)} = \int_0^1 u^{k-1} \left(\frac{1-up}{1-p}\right)^{n-k} du.$$

Since $\frac{1 - up}{1 - p}$ is increasing in p, so is $\frac{h(p)}{ph'(p)}$. Thus if a k out of n structure is composed of independent like components having an increasing failure rate, then the structure itself has an increasing failure rate.

If we note that the time of failure of a k out of n system corresponds to the k^{th} smallest in a sample of n observations, then an alternate statement of this result is the following:

Suppose $X_1 < X_2 < \cdots < X_n$ are a sample of order statistics based on independent observations from a distribution having increasing failure rate. Then the distribution of X_i has an increasing failure rate, $i = 1,2,\ldots,n$.

Actually we can generate new structures which have the property that $\frac{ph'(p)}{h(p)}$ is a decreasing function, by composition of structures having this property. Under composition we form a superstructure each element of which consists of copies of a given structure. If $h = f(g)$ with $g'(p) \geq 0$ then since

$$\frac{ph'(p)}{h(p)} = \frac{gf'(g)}{f(g)} \cdot \frac{pg'(p)}{g(p)} ,$$

the property is closed under composition.

3. Applications For systems composed of independent identical components each having an exponential failure distribution, the application of Theorem 1 is particularly simple:

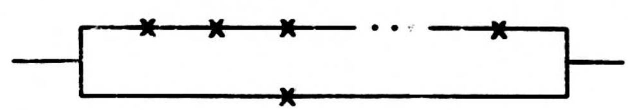
Theorem 2 Given a structure of like components with common failure distribution $F(t) = 1 - e^{-\lambda t}$. Then the failure rate $R(t)$ of the structure has as many changes of direction as a function of t as does $\frac{ph'(p)}{h(p)}$ as a function of p; moreover the changes occur in the opposite order. In particular, if $\frac{ph'(p)}{h(p)}$ is decreasing, then $R(t)$ is increasing.

Proof Since $r(t) = \lambda$, we have from Theorem 1 (a)

$$R(t) = \lambda \frac{ph'(p)}{h(p)}\bigg|_{p=1-F(t)}$$

Since $p = 1 - F(t)$ is a decreasing function of t, the conclusions follow. ||

From Theorem 2 we see that systems composed of like components having exponential failure distributions need not have monotonic failure rates. For example consider the following system composed of two subsystems in parallel, the first having k components in series, the second consisting of a single component. Assuming independent components, each having



exponential distribution for failure

$$F(t) = 1 - e^{-t},$$

we compute the probability $S(t)$ of system survival past time t to be

$$S(t) = 1 - (1 - e^{-t})(1 - e^{-kt}).$$

Thus

$$R(t) = -\frac{S'(t)}{S(t)} = \frac{e^{-t} + ke^{-kt} - (k+1)e^{-(k+1)t}}{e^{-t} + e^{-kt} - e^{-(k+1)t}} ,$$

and so

$$-\operatorname{sgn} R'(t) = \operatorname{sgn}[(k-1)^2 - k^2e^{-t} - e^{-kt}].$$

Note that given any integer $k > 1$, for $t = 0$, the sign is negative while for $t = \infty$, the sign is positive. Thus the system failure rate $R(t)$ is not monotonic for $k > 1$.

Next we use Theorem 1 to make an inference about the system structure from information about failure rates.

Theorem 3 Assume a system consists of a finite number of independent components, with reliability function $h(p)$ satisfying $h(1) = 1$. If $\frac{R(t)}{r(t)} \equiv k$, a constant, for all $0 \leq t < \infty$, then k must be an integer and the structure must consist of k components in series, with the remaining components, if any, non-essential*.

Proof $\frac{R(t)}{r(t)} \equiv k$ implies $\frac{ph'(p)}{h(p)} \equiv k$, by Theorem 1. Integrating, we have $h(p) = dp^k$, d a constant. Using the assumption $h(1) = 1$, we conclude d must be 1. Thus $h(p) = p^k$. Since the structure consists of a finite number of independent components, $h(p)$ must be a polynomial, and thus k must be an integer. The only finite structure of independent components having reliability function p^k is a series system of k components, with any additional components non-essential*. ||

*A non-essential component is one which may be omitted with no effect on structure performance.

Incidentally, the converse is immediate. That is, a structure having k independent components in series must have $\frac{R(t)}{r(t)} \equiv k$. Simply note that

$$R(t) = \frac{k\{1 - F(t)\}^{k-1} f(t)}{\{1 - F(t)\}^{k}} = kr(t),$$

where $F(t)$ is the component failure distribution.

4. <u>Bounds on System Failure Rate for Systems of Non-Identical Components</u>. Now let us consider a system of n independent components in which the i^{th} component has probability $p_i(t) = 1 - F_i(t)$ of still being operative at time t, $i = 1,2,\ldots,n$. If $h(\underline{p})$ is the structure reliability function where $\underline{p} = (p_1,\ldots,p_n)$, and $S(t)$ is the probability of system survival past time t, and $R(t)$ is the system failure rate, then since

$$S(t) = h(\underline{p}(t)),$$

we obtain

$$-\frac{dS}{dt} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\left(-\frac{dp_i}{dt}\right) = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \cdot f_i(t).$$

Hence

$$R(t) = \frac{-\frac{dS}{dt}}{S(t)} = \sum_{i=1}^{n} \frac{1 - F_i(t)}{S(t)} \cdot \frac{\partial h}{\partial p_i} \cdot \frac{f_i(t)}{1 - F_i(t)} = \sum_{i=1}^{n} \frac{p_i \frac{\partial h}{\partial p_i}}{h(\underline{p})} r_i(t). \qquad (4.1)$$

(4.1) states that system failure rate is the inner product of two vectors: (1) the vector whose i^{th} component is

$$\frac{p_i \partial h/\partial p_i}{h(\underline{p})},$$

and (2) the vector whose i^{th} component is $r_i(t)$. Note that the first vector is a function of the structure regardless of component failure rates, while the second vector is a function of component failure rates, regardless of structure.

Using (4.1) we may obtain bounds on system failure rate $R(t)$ as follows:

$$\left\{ \sum_{i=1}^{n} \frac{p_i \partial h/\partial p_i}{h(\underline{p})} \right\} \min_{1\leq i\leq n} r_i(t) \leq R(t) \leq \left\{ \sum_{i=1}^{n} \frac{p_i \partial h/\partial p_i}{h(\underline{p})} \right\} \max_{1\leq i\leq n} r_i(t) \qquad (4.2)$$

Next we shall show that for "k out of n" structures the factor $\sum_{i=1}^{n} \frac{p_i \partial h/\partial p_i}{h(\underline{p})}$ is strictly decreasing in each of $p_1,\ldots,p_n$.

<u>Theorem 4</u> Let $h(\underline{p})$ be the reliability function of a "k out of n" structure of independent components. Then

$$u(\underline{p}) = \sum_{i=1}^{n} \frac{p_i \partial h/\partial p_i}{h(\underline{p})}$$

is strictly decreasing in $0 \leq p_i \leq 1$, $i = 1,2,\ldots,n$.

<u>Proof</u> We may expand $h(\underline{p})$ as

$$h(\underline{p}) = p_i h(1_i,\underline{p}) + (1 - p_i)h(0_i,\underline{p}),$$

where $h(1_i,\underline{p})$ is the conditional reliability given the i^{th} component is operating, while $h(0_i,\underline{p})$ is the conditional reliability given the

i^{th} component has failed. It follows that

$$\frac{\partial h}{\partial p_i} = h(1_i, \underline{p}) - h(0_i, \underline{p}) = P[\sum_{j \neq i} X_j \geq k - 1] - P[\sum_{j \neq i} X_j \geq k]$$

where $X_j = 1$ or 0 according as the j^{th} component is operating or not. Hence

$$\frac{\partial h}{\partial p_i} = P[\sum_{j \neq i} X_j = k - 1].$$

Thus

$$\sum_{i=1}^{n} p_i \frac{\partial h}{\partial p_i} = \sum_{i=1}^{n} P[X_i = 1, \sum_{j \neq i} X_j = k - 1] = kP[\Sigma X_j = k]$$

since each term used in computing $\Sigma X_j = k$ will occur exactly k times in computing $\sum_{i=1}^{n} P[X_i = 1, \sum_{j \neq i} X_j = k - 1]$.

It will thus be sufficient to prove

$$\frac{P[\Sigma X_j = k]}{P[\Sigma X_j \geq k]}$$

is strictly decreasing in p_i, $i = 1,2,\ldots,n$, or equivalently, that

$$\frac{P[\Sigma X_j \geq k + 1]}{P[\Sigma X_j \geq k]}$$

is strictly increasing in p_i, $i = 1,2,\ldots,n$. This is equivalent to proving that for $p_i < p_i'$

$$0 > \begin{vmatrix} P[\Sigma X_j \geq k+1 | p_i] & P[\Sigma X_j \geq k+1 | p_i'] \\ P[\Sigma X_j \geq k | p_i] & P[\Sigma X_j \geq k | p_i'] \end{vmatrix}$$

$$= \begin{vmatrix} p_i P[\sum_{j\neq i} X_j \geq k] + q_i P[\sum_{j\neq i} X_j \geq k+1] & p_i' P[\sum_{j\neq i} X_j \geq k] + q_i' P[\sum_{j\neq i} X_j \geq k+1] \\ p_i P[\sum_{j\neq i} X_j \geq k-1] + q_i P[\sum_{j\neq i} X_j \geq k] & p_i' P[\sum_{j\neq i} X_j \geq k-1] + q_i' P[\sum_{j\neq i} X_j \geq k] \end{vmatrix}$$

$$= (p_i q_i' - p_i' q_i) \begin{vmatrix} P[\sum_{j\neq i} X_j \geq k] & P[\sum_{j\neq i} X_j \geq k+1] \\ P[\sum_{j\neq i} X_j \geq k-1] & P[\sum_{j\neq i} X_j \geq k] \end{vmatrix}$$

But $p_i q_i' - p_i' q_i < 0$. Also $P[\sum_{j\neq i} X_j = k]$ is totally positive of order infinity in differences of k [6] so that the determinant just above is non-negative; actually it can be proven positive. Thus the desired conclusion follows. ||

Thus for k out of n structures we may bracket the system failure rate $R(t)$ from below and above by a pair of bounds which are strictly increasing if component failure rates are increasing.

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